

HOSOYA POLYNOMIAL, WIENER AND HYPER-WIENER INDICES OF SOME REGULAR GRAPHS

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ABSTRACT

Let G be a graph. The distance $d(u,v)$ between two vertices u and v of G is equal to the length of a shortest path that connects u and v . The Wiener index $W(G)$ is the sum of all distances between vertices of G , whereas the hyper-Wiener index $WW(G)$ is defined as $WW(G) = \sum_{u,v \in V(G)} (d(v,u) + d(v,u)^2)$. Also, the Hosoya polynomial was introduced by H. Hosoya and define $H(G, x) = \sum_{\{u,v\} \in V(G)} x^{d(v,u)}$. In this paper, the Hosoya polynomial, Wiener index and Hyper-Wiener index of some regular graphs are determined.

KEYWORDS

Network Protocols, Topological distance, Hosoya polynomial, Wiener Index, Hyper-Wiener index, Regular graph, Harary graph.

1. INTRODUCTION

Let $G=(V;E)$ be a simple connected graph. The sets of vertices and edges of G are denoted by $V=V(G)$ and $E=E(G)$, respectively. The distance between vertices u and v of G , denoted by $d(u,v)$, is the number of edges in a shortest path connecting them. An edge $e=uv$ of graph G is joined between two vertices u and v ($d(u,v)=1$). The number of vertex pairs at unit distance equals the number of edges. Also, the topological diameter $D(G)$ is the longest topological distance in a graph G .

A topological index of a graph is a number related to that graph which is invariant under graph automorphism. The Wiener index $W(G)$ is the oldest topological indices, (based structure descriptors) [8], which have many applications and mathematical properties and defined by Harold Wiener in 1947 as:

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} \sum_{u \in V(G)} d(v,u) \quad (1.1)$$

Also, for this topological index, there is Hosoya Polynomial. The Hosoya polynomial was introduced by H. Hosoya, in 1988 [3] and define as follow:

$$H(G, x) = \frac{1}{2} \sum_{v \in V(G)} \sum_{u \in V(G)} x^{d(v,u)} \quad (1.2)$$

The Hosoya polynomial and Wiener index of some graphs computed [1, 2-4, 5]. Another topological index of graph (based structure descriptor) that was conceived somewhat later is the hyper-Wiener index that introduced by *Milan Randić* in 1993 [6] as

$$WW(G) = \frac{1}{2} \sum_{v \in V(G)} \sum_{u \in V(G)} (d(v,u) + d(v,u)^2) \quad (1.3)$$

$$= \frac{1}{2} W(G) + \frac{1}{2} \sum_{v \in V(G)} \sum_{u \in V(G)} d(v,u)^2 \quad (1.4)$$

In this paper, we obtained a closed formula of the Hosoya polynomial, Wiener and Hyper-Wiener indices for an interesting regular graph that called *Harary graph*. The general form of the Harary graph $H_{2m,n}$ is defined as follows:

Definition 1.1 [7]. Let m and n be two positive integer numbers, then the Harary graph $H_{2m,n}$ is constructed as follows:

It has vertices $1, 2, \dots, n-1, n$ and two vertices i and j are joined if $i-m \leq j \leq i+m$ (where addition is taken modulo n).

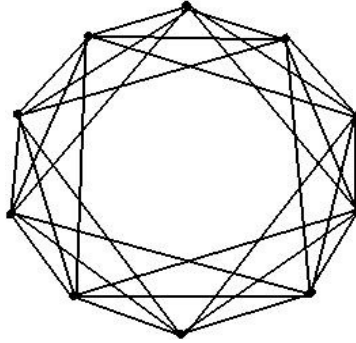


Figure1. The Harary graph $H_{6,10}$.

2. MAIN RESULTS

In this section we compute the Hosoya polynomial, Wiener index and hyper-Wiener index of the Harary graph $H_{2m,n}$.

Theorem 2.1. Consider the Harary graph $H_{2m,n}$ for all positive integer number m and n . Then,

- The Hosoya polynomial of $H_{2m,n}$ for n odd is

$$H_{2m,n} = \sum_{d=1}^{\lfloor n/2m \rfloor} mnx^d + n \left(\left\lfloor \frac{n}{2} \right\rfloor - m \times \left\lfloor \frac{n}{2m} \right\rfloor \right) x^{\lfloor n/2m \rfloor + 1}$$

- The Hosoya polynomial of $H_{2m,n}$ for n even ($=2q$) is

$$H_{2m,2q} = \sum_{d=1}^{\lfloor q/m \rfloor} 2qmx^d + 2q^2 - 2qm \left\lfloor \frac{q}{m} \right\rfloor - qx^{\lfloor q/m \rfloor + 1}$$

Proof. Let H be the Harary graph $H_{2m,n}$ (Figure 1). The number of vertices in this graph is equal to $|V(H_{2m,n})|=n$ ($\forall m,n \in \mathbb{N}$). Also, two vertices $v_i, v_j \in V(H_{2m,n})$ are adjacent if and only if $|i-j| \leq m$, then $d(v_i, v_j)=1$ thus the number of edges of this regular graph is $|E(H_{2m,n})| = \frac{n(2m)}{2} = mn$.

It is obvious that for all graph G , $H(G, 1) = \sum_{i=1}^{d(G)} d(G, i) = \binom{n}{2} = \frac{n(n-1)}{2}$. On the other hand from the structure of $H_{2m,n}$ (Figure 1), we see that all vertices of $H_{2m,n}$ have similar geometrical and topological conditions then the number of path as distance i in $H_{2m,n}$ ($d(H_{2m,n}, i)$) is a multiple of the number of vertices (n). In other words, any vertex $v \in V(H_{2m,n})$ is an endpoint of $d(H_{2m,n}, i)/n$ paths as distance i in $H_{2m,n}$ ($=d_v(H_{2m,n}, i)$). For example $\forall v \in V(H_{2m,n})$, there are $2m = d_v(H_{2m,n}, 1)$ paths as distance 1 (or all edges incident to v) in $H_{2m,n}$. From Figure 1, one can see that $\forall i=1, 2, \dots, n$ and $v_i, v_{i+m} \in V(H_{2m,n})$, $d(v_i, v_{i \pm m + j}) = 2$ ($j=\{1, 2, \dots, m\}$) because $v_i, v_{i \pm m}, v_{i \pm m + j} \in E(H_{2m,n})$ and $d(v_i, v_{i \pm m}) = d(v_{i \pm m}, v_{i \pm m + j}) = 1$. Obviously $d_{v_i}(H_{2m,n}, 2) = 2m$

Now, since $\forall i \in \{1, 2, \dots, n\}$ and $v_i, v_{i+m} \in V(H_{2m,n})$, $d(v_i, v_{i \pm m}) = d(v_{i \pm m}, v_{i \pm 2m}) = d(v_{i \pm 2m}, v_{i \pm 3m}) = \dots = d(v_{i \pm (k-1)m}, v_{i \pm km}) = 1$ such that $k \leq \lfloor n/2m \rfloor$. Thus $\forall d=1, \dots, k$ $d(v_i, v_{i \pm dm}) = d$ and obviously $d(v_i, v_{i \pm dm + j}) = d+1$ ($j=1, 2, \dots, m-1$). By these mentions, one can see that for an arbitrary vertex v_i the diameter $D(H_{2m,n})$ of this Harary graph is $d(v_i, v_{i + \lfloor n/2 \rfloor}) = \lfloor n/2m \rfloor + 1$. Also the distance between vertices v_i and $v_{i \pm \lfloor (n/2r)m + j \rfloor}$ is equal to $D(H_{2m,n})$ for all $j=1, 2, \dots, \lfloor n/2 \rfloor$. It's easy to see that if n be odd then $\forall v_i \in V(H_{2m,n})$, $d_{v_i}(H, \lfloor n/2m \rfloor + 1) = \lfloor n/2 \rfloor - m \times \lfloor n/2m \rfloor$, else n be even then $d(H, d(H)) = \frac{n}{2} (\lfloor \frac{n}{2} \rfloor - m \lfloor \frac{n}{2m} \rfloor - 1)$.

Now by using the definition of Hosoya polynomial (equation 2), we have

$$\begin{aligned} H(H_{2m,n}, x) &= \sum_{\{u,v\} \in V(H_{2m,n})} x^{d(v,u)} \\ &= \sum_{i,j=1}^n x^{d(v_i, v_j)} \\ &= \sum_{i=1}^n \sum_{j=0}^{m-1} \sum_{k=0}^{\lfloor \frac{n}{2m} \rfloor} x^{d(v_i, v_{i \pm k + j})} \\ &= mn x^1 + mn x^2 + \dots + mn x^{\lfloor \frac{n}{2m} \rfloor} + d(H_{2m,n}, \lfloor \frac{n}{2m} \rfloor + 1) x^{\lfloor \frac{n}{2m} \rfloor + 1} \end{aligned}$$

$$\text{where } d(H_{2m,n}, \lfloor \frac{n}{2m} \rfloor + 1) = \begin{cases} n \times \left\lfloor m \times \left\lfloor \frac{n}{2m} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor & n \text{ odd} \\ n \times \left\lfloor m \times \left\lfloor \frac{n}{2m} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor - \frac{n}{2} & n \text{ even} \end{cases}$$

Here the proof of Theorem 2.1 is completed.

Theorem 2.2. Let $H_{2m,n}$ be the Harary graph. Then the Wiener index of $H_{2m,n}$ is equal to

- If n be odd, $W(H_{2m,2q+1}) = (2q+1) \left(q + \left\lfloor \frac{q}{m} \right\rfloor \left(q - \frac{m}{2} \right) - \frac{m}{2} \left\lfloor \frac{q}{m} \right\rfloor^2 \right)$
- If n be even, $W(H_{2m,2q}) = (2q^2 - q) + q \left\lfloor \frac{q}{m} \right\rfloor (2q - 1 - m) - qm \left\lfloor \frac{q}{m} \right\rfloor^2$

Proof. By according to the definitions of the Wiener index and Hosoya Polynomial of a graph G (Equation 1.1 and 1.2), then the Wiener index $W(G)$ will be the first derivative of Hosoya polynomial $H(G,x)$ evaluated at $x=1$. Thus by using the results from Theorem 2.1, we have

1- If n be an arbitrary even positive integer number ($n=2q$), then

$$\begin{aligned} W(H_{2m,2q}) &= \sum_{\{u,v\} \in V(H_{2m,2q})} d(v,u) \\ &= \left. \frac{\partial H(H_{2m,2q}, x)}{\partial x} \right|_{x=1} \\ &= \left. \frac{\partial \sum_{d=1}^{\lfloor q/m \rfloor} 2qmx^d + 2q^2 - 2qm \left\lfloor \frac{q}{m} \right\rfloor - q \cdot x^{\lfloor q/m \rfloor + 1}}{\partial x} \right|_{x=1} \\ &= 2qm \sum_{d=1}^{\lfloor q/m \rfloor} d + \left(-2qm \left\lfloor \frac{q}{m} \right\rfloor^2 + \left\lfloor \frac{q}{m} \right\rfloor (2q^2 - q - 2qm) + (2q^2 - q) \right) \\ &= (2q^2 - q) + q(2q - 1 - m) \left\lfloor \frac{q}{m} \right\rfloor - qm \left\lfloor \frac{q}{m} \right\rfloor^2 \end{aligned}$$

2- If n be an odd positive integer number ($n=2q+1$), then

$$\begin{aligned} W(H_{2m,2q+1}) &= \left. \frac{\partial \sum_{d=1}^{\lfloor n/2m \rfloor} mn x^d + n \left(\left\lfloor \frac{n}{2} \right\rfloor - m \times \left\lfloor \frac{n}{2m} \right\rfloor \right) x^{\lfloor n/2m \rfloor + 1}}{\partial x} \right|_{x=1} \\ &= (2q+1) \left(\sum_{d=1}^{\lfloor n/2m \rfloor} m \times d + \left(q - m \left\lfloor \frac{q}{m} \right\rfloor \right) \left(\left\lfloor \frac{q}{m} \right\rfloor + 1 \right) \right) \\ &= (2q^2 + q) + \left\lfloor \frac{q}{m} \right\rfloor \left(2q^2 + q - \frac{m(2q+1)}{2} \right) - \frac{m(2q+1)}{2} \left\lfloor \frac{q}{m} \right\rfloor^2. \end{aligned}$$

Corollary 2.3. Suppose n be an even positive integer number ($n=2q$) and $2m|n$, then such that $q/m=r$. Thus $W(H_{2m,2mr})=mr(mr^2+mr-r-1)$.

Theorem 2.4. The hyper-Wiener index of the Harary graphs $H_{2m,2q}$ and $H_{2m,2q+1}$ are equal to

$$\begin{aligned} WW(H_{2m,2q}) &= \frac{3}{2} (2q^2 - q) + \left(5q^2 - \frac{5}{2}q - \frac{13}{6}qm \right) \left\lfloor \frac{q}{m} \right\rfloor + \left(2q^2 - q - \frac{7}{2}qm \right) \left\lfloor \frac{q}{m} \right\rfloor^2 - \frac{2}{3}mq \left\lfloor \frac{q}{m} \right\rfloor^3 \\ WW(H_{2m,2q+1}) &= (2q+1) \left[-\frac{2}{3}m \left\lfloor \frac{q}{m} \right\rfloor^3 + \left(q - \frac{7}{4}m \right) \left\lfloor \frac{q}{m} \right\rfloor^2 + \left(\frac{5}{2}q - \frac{13}{12}m \right) \left\lfloor \frac{q}{m} \right\rfloor + \frac{3}{2}q \right] \end{aligned}$$

Proof. Consider the Harary graph $H_{2m,n}$ and refer to Equations 1.3 and 1.4. Thus

$$WW(H_{2m,n}) = \frac{1}{2} W(H_{2m,n}) + WW^*(H_{2m,n})$$

$$\text{where } WW^*(H_{2m,n}) = \sum_{\{u,v\} \in V(H_{2m,n})} d^2(v,u)$$

Now, suppose n be even ($n=2q$), therefore

$$WW^*(H_{2m,2q}) = 2qm \sum_{d=1}^{\lfloor q/m \rfloor} d^2 + \left(2q^2 - q - 2qm \left\lfloor \frac{q}{m} \right\rfloor \right) \left(\left\lfloor \frac{q}{m} \right\rfloor + 1 \right)^2$$

$$= \frac{2qm}{6} \left(2 \left[\frac{q}{m} \right]^3 + 3 \left[\frac{q}{m} \right]^2 + \left[\frac{q}{m} \right] \right) + \left(2q^2 - q - 2qm \left[\frac{q}{m} \right] \right) \left(\left[\frac{q}{m} \right]^2 + 2 \left[\frac{q}{m} \right] + 1 \right)$$

$$= (2q^2 - q) + \left(4q^2 - 2q - \frac{5}{3}qm \right) \left[\frac{q}{m} \right] + (2q^2 - q - 3qm) \left[\frac{q}{m} \right]^2 - \frac{2}{3}mq \left[\frac{q}{m} \right]^3$$

Thus, $WW(H_{2m,2q}) = \frac{3}{2}(2q^2 - q) + \left(5q^2 - \frac{5}{2}q - \frac{13}{6}qm \right) \left[\frac{q}{m} \right] + \left(2q^2 - q - \frac{7}{2}qm \right) \left[\frac{q}{m} \right]^2 - \frac{2}{3}mq \left[\frac{q}{m} \right]^3$

If n be odd ($n=2q+1$), therefore

$$WW^*(H_{2m,2q+1}) = m(2q+1) \sum_{d=1}^{\left[\frac{q}{m} \right]} d^2 + (2q+1) \left(q - m \left[\frac{q}{m} \right] \right) \left(\left[\frac{q}{m} \right] + 1 \right)^2$$

$$= \frac{(2q+1)m}{6} \left(2 \left[\frac{q}{m} \right]^3 + 3 \left[\frac{q}{m} \right]^2 + \left[\frac{q}{m} \right] \right) + (2q+1) \left(q - m \left[\frac{q}{m} \right] \right) \left(\left[\frac{q}{m} \right]^2 + 2 \left[\frac{q}{m} \right] + 1 \right)$$

$$= (2q+1) \left[-\frac{2}{3}m \left[\frac{q}{m} \right]^3 + \left(q - \frac{3}{2}m \right) \left[\frac{q}{m} \right]^2 + \left(2q - \frac{5}{6}m \right) \left[\frac{q}{m} \right] + q \right]$$

Thus, $WW(H_{2m,2q+1}) = (2q+1) \left[-\frac{2}{3}m \left[\frac{q}{m} \right]^3 + \left(q - \frac{7}{4}m \right) \left[\frac{q}{m} \right]^2 + \left(\frac{5}{2}q - \frac{13}{12}m \right) \left[\frac{q}{m} \right] + \frac{3}{2}q \right]$.

And theses complete the proof of Theorem 2.4.

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